Lecture 23: More on Determinants

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21.4 Useful properties of determinants

Theorem

Let A and B be nxn matrices.
Then,

i) $\det(rA) = r^n \cdot \det(A)$

$$\text{ie. } \det\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = 1 \quad \Rightarrow \quad \det\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = -1$$

$$\Rightarrow \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-1)\cdot(-1) = 1$$

ii) $\det(AB) = \det(A) \cdot \det(B)$

iii) $\det(A) \neq 0 \Rightarrow A \text{ is invertible}$

in this case

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Application

$A = \begin{bmatrix} x & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 3 \end{bmatrix}$. Find all $x \in \mathbb{R}$ such that A is not invertible.

(**If A is not invertible, the determinant must equal 0**)

expand along 1st row

$$\det(A) = x \cdot \det\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} + 1 \cdot \det\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = 6x - 2$$

$$\det(A) = 0 \;\; \boldsymbol{iff} \;\; 6x - 2 = 0 \;\; \boldsymbol{iff} \;\; x = \frac{1}{3}$$

Answer: For $x = \frac{1}{3}$ the matrix is not invertible.

22 Eigenvalues and Eigenvectors

$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$

$A \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ irrelevant

$A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \end{bmatrix} = 4 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ good

$A \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

21.1

Definition

Let A be an nxn matrix.

If $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n, x \neq 0$
such that:

$$Ax = \lambda x$$

then λ is called an **eigenvalue** of A and x is called an **eigenvector** of A.

22.2 How to find the eigenvalues

We want to find all $\lambda \in \mathbb{R}$ such that there is a non-trivial (non-zero) vector $x \in \mathbb{R}^n$ with:

$$Ax = \lambda x \;\; \Rightarrow \;\; Ax = \lambda I x \quad \text{identity matrix}$$

$$\Rightarrow Ax - \lambda I x = 0$$

$$\Rightarrow (A - \lambda I)x = 0$$

$$\Rightarrow (A-\lambda I)x = 0$$

This is the case **iff** the homogeneous system $(A-\lambda I)x = 0$ has a non-trivial solution.
This holds **iff** $A - \lambda I$ is not invertible.
This holds **iff** $\det(A-\lambda I) = 0$.

$\det(A-\lambda I)$ is called the **characteristic polynomial** of A.

So, λ is an eigenvalue **iff** $\det(A-\lambda I) = 0$.

Example

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix} \qquad A - \lambda I = \begin{bmatrix} 3-\lambda & -1 \\ -2 & 2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A-\lambda I) &= (3-\lambda)\cdot(2-\lambda) = 2 \\ &= \lambda^2 - 5\lambda + 4 \\ &= (\lambda-1)(\lambda-4) \end{aligned}$$

$\lambda = 1$ and $\lambda = 4$ are the only roots,
hence the only **eigenvalues**.
Next step: Find corresponding **eigenvectors**.

22.3 How to find the corresponding eigenvectors

Eigenvectors are the non-trivial solutions to:

$$(A-\lambda I)x = 0$$

So, we need to compute the null space/kernel of $A - \lambda I$.

Definition

Let λ be an eigenvalue of A. Then the subspace:

$$E_\lambda = \ker(A-\lambda I)$$

is called the λ-eigenspace. Its non-trivial elements are eigenvectors corresponding to A.

In our example:

$$A = \begin{bmatrix} 3 & -1 \\ -2 & 2 \end{bmatrix}$$

$\lambda = 1$

$$\begin{aligned} E_\lambda = \ker(A-\lambda I) &= \ker \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \\ &= \left[\begin{array}{cc|c} 2 & -1 & 0 \\ -2 & 1 & 0 \end{array}\right] \\ &\sim \left[\begin{array}{cc|c} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{array}\right] \end{aligned}$$

$$S = \left\{ s \cdot \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$$

In particular, $\begin{pmatrix} 1 \\ 2 \end{pmatrix} \in E_1$

$\lambda = 4$

$$E_4 = \ker(A - 4I) = \ker \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 & 0 \\ -2 & -2 & 0 \end{bmatrix} \sim \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{RREF}}$$

$$S = \left\{ \begin{pmatrix} -s \\ s \end{pmatrix} \middle| s \in \mathbb{R} \right\} = \left\{ s \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} \middle| s \in \mathbb{R} \right\}$$

In particular, $\begin{pmatrix} -1 \\ 1 \end{pmatrix} \in E_4$

In our example:

$\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ basis of E_1

$\left\{\begin{pmatrix}-1\\1\end{pmatrix}\right\}$ basis of E_4

$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$ happens to be a basis of $\mathbb{R}^2$.

In this case, A is diagonazible.